Semiclassical strings in $AdS_5 \times S^5$ and scalar operators in $\mathcal{N} = 4$ SYM theory

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Abstract

We review recent progress in quantitative checking of AdS/CFT duality in the sector of "semiclassical" string states dual to "long" scalar operators of $\mathcal{N} = 4$ super Yang-Mills theory. In particular, we discuss the effective action approach, in which the same sigma model type action describing coherent states is shown to emerge from the $AdS_5 \times S^5$ string action and from the integrable spin chain Hamiltonian representing the SYM dilatation operator.

1 Introduction

The $\mathcal{N} = 4$ SYM theory is a remarkable example of 4-d conformal field theory. In the planar ($N \to \infty$) limit it is parametrized by the 't Hooft coupling $\lambda = g^2_{\rm YM} N$, and the major first step towards the solution of this theory would be to determine the spectrum of anomalous dimensions $\Delta(\lambda)$ of the primary operators built out of products of local gauge-covariant fields. That this may be possible in principle is suggested by the AdS/CFT duality implying the existence of hidden integrable 2-d structure corresponding to $AdS_5 \times S^5$ string sigma model.

The AdS/CFT duality implies the equality between the AdS energies of quantum closed string states as functions of the effective string tension $T = \frac{\sqrt{\lambda}}{2\pi}$ and quantum numbers like S^5 angular momenta $Q = (J, ...)$ and dimensions of the corresponding local SYM operators. To give a quantitative check of the duality one would like to understand how strings "emerge" from the field theory, in particular, which (local, single-trace) gauge theory operators [1] correspond to which "excited" string states and how one may verify

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the matching of their dimensions/energies beyond the well-understood BPS/supergravity sector. We would like to use the duality as a guide to deeper understanding of the structure of quantum SYM theory. In particular, results motivated by comparison to string theory may allow one to “guess” the general structure of the SYM anomalous dimension matrix and may also suggest new methods of computing anomalous dimensions in less supersymmetric gauge theories.

Below we shall review recent progress in checking AdS/CFT correspondence in a subsector of string/SYM states with large quantum numbers. Let us start with brief remarks on SYM and string sides of the duality. The SYM theory contains a gauge field, 6 scalars ϕ_m and 4 Weyl fermions, all in adjoint representation of $SU(N)$. It has global conformal and R-symmetry, i.e. is invariant under $SO(2,4) \times SO(6)$. To determine (in planar limit) dimensions of local gauge-invariant operators one in general needs to find the anomalous dimension matrix to all orders in λ and then to diagonalize it. The special case is that of chiral primary or BPS operators (and their descendants) $\mathrm{tr}(\phi_{\{m_1} ... \phi_{m_k\}})$ whose dimensions are protected, i.e. do not depend on λ. The problem of finding dimensions appears to simplify also in the case of “long” operators containing large number of fields under the trace. One example is provided by “near-BPS” operators [2] like $\mathrm{tr}(\Phi_1^J \Phi_2^n ...) + ...$ where $J \gg n$, and $\Phi_k = \phi_k + i\phi_{k+3}, \ \ k = 1, 2, 3$. Below we will consider “far-from-BPS” operators like $\mathrm{tr}(\Phi_1^{J_1} \Phi_2^{J_2} ...) + ...$ where $J_1 \sim J_2 \gg 1$.

The type IIB string action in $AdS_5 \times S^5$ space has the following structure

$$I = -\frac{1}{2} T \int d\tau \int_0^{2\pi} d\sigma \, (\partial^p Y^\mu \partial_p Y^\nu \eta_{\mu\nu} + \partial^p X^m \partial_p X^n \delta_{mn} + ...) \ , \tag{1}$$

where $Y^\mu Y^\nu \eta_{\mu\nu} = -1, \ \ X^m X^n \delta_{mn} = 1$, $\eta_{\mu\nu} = (-++++-)$, $T = \frac{\sqrt{\lambda}}{2\pi}$ and dots stand for the fermionic terms [3] that ensure that this model defines a 2-d conformal field theory. The closed string states can be classified by the values of the Cartan charges of the obvious symmetry group $SO(2,4) \times SO(6)$, i.e. $(E, S_1, S_2; J_1, J_2, J_3)$, i.e. by the AdS_5 energy, two spins in AdS_5 and 3 spins in S^5. The mass shell condition gives a relation $E = E(Q,T)$. Here T is the string tension and $Q = (S_1, S_2, J_1, J_2, J_3; n_k)$ where n_k stand for higher conserved charges (analogs of oscillation numbers in flat space).

According to AdS/CFT duality quantum closed string states in $AdS_5 \times S^5$ should be dual to quantum SYM states at the boundary $R \times S^3$ or, via radial quantization, to local single-trace operators at the origin of R^4. The energy of a string state should then be equal to the dimension of the corresponding SYM operator, $E(Q,T) = \Delta(Q, \lambda)$, where on the SYM side the charges Q characterise the operator. By analogy with flat space and ignoring α' corrections (i.e. assuming $R \to \infty$ or $\alpha' \to 0$) the excited string states are

expected to have energies $E \sim \frac{1}{\sqrt{\alpha'}} \sim \lambda^{1/4}$ [4] which represents a non-trivial prediction for strong-coupling asymptotics of SYM dimensions. In general, the natural (inverse-tension) perturbative expansion on the string side will be given by $\sum_n \frac{c_n}{(\sqrt{\lambda})^n}$, while on the SYM side the usual planar perturbation theory will give the eigenvalues of the anomalous dimension matrix as $\Delta = \sum_n a_n \lambda^n$. The AdS/CFT duality implies that the two expansions are to be the strong-coupling and weak-coupling asymptotics of the same function. To check the relation $E = \Delta$ is then a non-trivial problem. On the symmetry grounds, this can be shown in the case of 1/2 BPS (chiral primary) operators dual to supergravity states ("massless" or ground state string modes) since their energies/dimensions are protected from corrections.

For generic non-BPS states the situation looked hopeless until the remarkable suggestion [2, 4] that a progress in checking duality can be made by concentrating on a subsector of states with large ("semiclassical") values of quantum numbers, $Q \sim T \sim \sqrt{\lambda}$ (here Q stands for generic quantum number like spin in AdS_5 or S^5 or an oscillation number) and considering a new limit

$$Q \to \infty , \qquad \tilde{\lambda} \equiv \frac{\lambda}{Q^2} = \text{fixed} . \tag{2}$$

On the string side $\frac{Q}{\sqrt{\lambda}} = \frac{1}{\sqrt{\tilde{\lambda}}}$ plays the role of the semiclassical parameter (like rotation frequency) which can then be taken to be large. The energy of such states happens to be $E = Q + f(Q, \lambda)$. The duality implies that such semiclassical string states as well as near-by fluctuations should be dual to "long" SYM operators with large canonical dimension, i.e. containing large number of fields or derivatives under the trace. In this case the duality map becomes more explicit.

The simplest possibility is to start with a BPS state that carries a large quantum number and consider small fluctuations near it [2], i.e. a set of *near-BPS* states characterised by a large parameter [2]. The only non-trivial example of such BPS state is represented by a point-like string moving along a geodesic in S^5 with a large angular momentum $Q = J$. Then $E = J$ and the dual operator is tr Φ^J, $\Phi = \phi_1 + i\phi_2$. The small (nearly point-like) closed strings representing near-by fluctuations are ultrarelativistic, i.e. their kinetic energy is much larger than their mass. They are dual to SYM operators of the form $\mathrm{tr}(\Phi^J ...)$ where dots stand for a small number of other fields and/or covariant derivatives (and one needs to sum over different orders of the factors to find an eigenstate of the anomalous dimension matrix). The energies of the small fluctuations happen to be [5, 2] $E = J + \sqrt{1 + n^2 \tilde{\lambda}}\ N_n + O(\frac{1}{J})$. One can argue in general [6, 7] and check explicitly [8, 9] that higher-order quantum string sigma model corrections are indeed suppressed in the limit (2), i.e. in the large J, fixed $\tilde{\lambda} = \frac{\lambda}{J^2} \equiv \lambda'$ limit. The remarkable feature of

this expression is that E is analytic in $\tilde{\lambda}$, suggesting direct comparison with perturbative SYM expansion in λ. Indeed, it can be shown that the first two $\tilde{\lambda}$ and $\tilde{\lambda}^2$ terms in the expansion of the square root agree precisely with the one [2] and two [10] (and also three [11, 12]) loop terms in the anomalous dimensions of the corresponding operators. There is also a interesting argument [13] (for a 2-impurity case) suggesting how the full $\sqrt{1+n^2\tilde{\lambda}}$ expression can appear on the perturbative SYM side. However, the general proof of the consistency of the BMN limit on the SYM side (i.e. that the usual perturbative expansion can be rewritten as an expansion in $\tilde{\lambda}$ and $\frac{1}{J}$) remains to be given; also, to explain why the string and SYM expressions match one should show that the string limit (first $J \to \infty$, then $\tilde{\lambda} = \frac{\lambda}{J^2} \to 0$) and the SYM limit (first $\lambda \to 0$, then $J \to \infty$) produce the same expressions for the dimensions (cf. [14, 15, 16]).

If one moves away from the near-BPS limit and considers, e.g., a non-supersymmetric state with a large angular momentum $Q = S$ in AdS_5 [4], a direct quantitative check of the duality is no longer possible: here the classical energy is not analytic in λ and quantum corrections are no longer suppressed by powers of $\frac{1}{S}$. However, it is still possible to demonstrate a remarkable qualitative agreement between S-dependence of the string energy and SYM anomalous dimension. The energy of a folded closed string rotating at the center of AdS_5 which is dual to the twist 2 operators on the SYM side ($\mathrm{tr}(\Phi_k^* D^S \Phi_k)$, $D = D_1 + iD_2$ and similar operators with spinors and gauge bosons that mix at higher loops [19, 20]) has the form (when expanded at large S): $E = S + f(\lambda)\ln S + ...$ On the string side

$$f(\lambda)_{_{\lambda\gg 1}} = c_0\sqrt{\lambda} + c_1 + \frac{c_2}{\sqrt{\lambda}} + ... \ ,$$

where $c_0 = \frac{1}{\pi}$ is the classical [4] and $c_1 = -\frac{3}{\pi}\ln 2$ is the 1-loop [6] coefficient. On the gauge theory side one finds the *same* S-dependence of the anomalous dimension with the perturbative expansion of the $\ln S$ coefficient being

$$f(\lambda)_{_{\lambda\ll 1}} = a_1\lambda + a_2\lambda^2 + a_3\lambda^3 + ... \ ,$$

where $a_1 = \frac{1}{2\pi^2}$ [18], $a_2 = -\frac{1}{96\pi^2}$ [19], and $a_3 = \frac{11}{360\times 64\pi^2}$ [20]. Like in the case of the SYM entropy [21], here one expects the existence of a smooth interpolating function $f(\lambda)$ that connects the two perturbative expansions (indeed, a simple square root formula seems to give a good fit [19, 20]).

One could wonder still if examples of quantitative agreement between string energies and SYM dimensions observed for near-BPS (BMN) states can be found also for more general non-BPS string states. Indeed, it was noticed already in [6] that a string state that carries large spin in AdS_5 as well as large spin $J = 0$ in S^5 has, in contrast to the

above $J = 0$ case, an analytic expansion of its energy in $\tilde{\lambda} = \frac{\lambda}{J^2}$, just as in the BMN case with $N_n \sim S$. It was observed in [22] that semiclassical string states carrying several large spins (with at least one of them being in S^5) have regular expansion of their energy E in powers of $\tilde{\lambda}$ and it was suggested, by analogy with the near-BPS case, that E can be matched with perturbative expansion for the SYM dimensions.

For a classical rotating closed string solution in S^5 one has $E = \sqrt{\lambda}\mathcal{E}(w_i), \ J_i = \sqrt{\lambda} w_i$ so that $E = E(J_i, \lambda)$ and the key property is that there is no $\sqrt{\lambda}$ factors in E (in contrast to the case of a single spin in AdS_5)

$$E = J + c_1 \frac{\lambda}{J} + c_2 \frac{\lambda^2}{J^3} + ... = J \ \left[1 + c_1 \tilde{\lambda} + c_2 \tilde{\lambda}^2 + ...\right] \ . \tag{3}$$

Here $J = \sum_{i=1}^3 J_i, \ \tilde{\lambda} \equiv \frac{\lambda}{J^2}$ and $c_n = c_n(\frac{J_i}{J})$ are functions of ratios of the spins which are finite in the limit $J_i \gg 1, \ \tilde{\lambda}$ =fixed. The simplest example of such a solution is provided by a circular string rotating in two orthogonal planes in S^3 part of S^5 with the two angular momenta being equal $J_1 = J_2$ [22]: $\mathrm{X}_1 \equiv X_1 + iX_2 = \cos(n\sigma) \ e^{iw\tau}, \ \ \mathrm{X}_2 \equiv X_3 + iX_4 = \sin(n\sigma) \ e^{iw\tau}$, with the global AdS_5 time being $t = \kappa\tau$. $(Y_5 + iY_0 = e^{it})$. The conformal gauge constraint implies $\kappa^2 = w^2 + n^2$ and thus $E = \sqrt{J^2 + n^2\lambda}$ or $E = J(1 + \frac{1}{2}n^2\tilde{\lambda} - \frac{1}{8}n^4\tilde{\lambda}^2 + ...)$, where $J = J_1 + J_2 = 2J_1$. For fixed J the energy thus has a regular expansion in tension (in contrast to what happens in flat space where $E = \sqrt{\frac{2}{\alpha'}J}$). Similar expressions (3) are found also for more general multispin closed strings [22, 23, 24, 25, 26, 27]. In particular, for a folded string string rotating in one plane of S^5 and with its center of mass orbiting along big circle in another plane [24] the coefficients c_n are transcendental functions (expressed in terms of elliptic integrals). More generally, the 3-spin solutions are described by an integrable Neumann model [25, 26] and the coefficients c_n in the energy are expressed in terms of genus two hyperelliptic functions.

To be able to compare the classical energy to the SYM dimension one should be sure that string α' corrections are suppressed in the limit $J \to \infty, \ \tilde{\lambda}$ =fixed. Formally, this should be the case since $\alpha' \sim \frac{1}{\sqrt{\lambda}} \sim \frac{1}{J\sqrt{\tilde{\lambda}}}$, but, what is more important, the $\frac{1}{J}$ corrections are again analytic in $\tilde{\lambda}$ [23], i.e. the expansion in large J and small $\tilde{\lambda}$ is well-defined on the string side,

$$E = J \left[1 + \tilde{\lambda}(c_1 + \frac{d_1}{J} + ...) + \tilde{\lambda}^2(c_2 + \frac{d_2}{J} + ...) + ...\right] \ , \tag{4}$$

with the classical energy (3) being the $J \to \infty$ limit of the exact expression. The reason for this particular form of the energy (4) can be explained as follows [7, 27]: we are computing string sigma model loop corrections to the mass of a stationary solitonic solution on a

2-d cylinder (no IR divergences). This theory is conformal (due to the crucial presence of fermionic fluctuations) and thus does not depend on UV cutoff. The relevant fluctuations are massive and their masses scale as $w \sim \frac{1}{\sqrt{\tilde{\lambda}}}$. As a result, the inverse mass expansion is well-defined and the quantum corrections should be proportional to positive powers of $\tilde{\lambda}$.

Similar expressions are found for the energies of small fluctuations near a given classical solution: as in the BMN case, the fluctuation energies are suppressed by extra factor of J, i.e. $\delta E = \tilde{\lambda}(k_1 + \frac{m_1}{J} + ...) + \tilde{\lambda}^2(k_2 + \frac{m_2}{J} + ...) + ...$

Assuming that the same limit is well-defined also on the SYM side, one should then be able to compare the coefficients in (4) to the coefficients in the anomalous dimensions of the corresponding SYM operators [22] $\mathrm{tr}(\Phi_1^{J_1}\Phi_2^{J_2}\Phi_3^{J_3}) + ...$ (and also do similar matching for near-by fluctuation modes). In practice, it is known at least in principle how to compute the dimensions in a different limit: first expanding in λ and then expanding in $\frac{1}{J}$. One may expect that this expansion of anomalous dimensions takes the form equivalent to (4), i.e.

$$\Delta = J + \lambda(\frac{a_1}{J} + \frac{b_1}{J^2} + ...) + \lambda^2(\frac{a_2}{J^3} + \frac{b_2}{J^4} + ...) + ... \ , \tag{5}$$

and that the respective coefficients in E and Δ agree with each other. The subsequent work [28, 29, 30, 31, 32, 33, 15, 34, 35, 36] did verify this structure of Δ and moreover established the general agreement between the two leading coefficients c_1, c_2 in E (4) and the "one-loop" and "two-loop" coefficients a_1, a_2 in Δ.

To compute Δ one is first to diagonalize anomalous dimension matrix defined on a set of long scalar operators. The crucial step was made in [28] where it was observed that the one-loop planar dilatation operator in the scalar sector can be interpreted as a Hamiltonian of an integrable $SO(6)$ spin chain and thus can be diagonalized even for large length $L = J$ by the Bethe ansatz method. In the simplest case of (closed) "$SU(2)$" sector of operators $\mathrm{tr}(\Phi_1^{J_1}\Phi_2^{J_2}) + ...$ built out of two chiral scalars the dilatation operator can be interpreted as "spin up" and "spin down" states of periodic $\mathrm{XXX}_{1/2}$ spin chain with length $L = J = J_1 + J_2$. Then the 1-loop dilatation operator becomes equivalent to the Hamiltonian of the ferromagnetic Heisenberg model

$$D_1 = \frac{\lambda}{(4\pi)^2} \sum_{l=1}^{J} (1 - \vec{\sigma}_l \cdot \vec{\sigma}_{l+1}) \ . \tag{6}$$

By considering the thermodynamic limit ($J \to \infty$) of the Bethe ansatz the proposal of [22] was confirmed at the leading order of expansion in $\tilde{\lambda}$ [29, 30]: for eigen-operators with $J_1 \sim J_2 \gg 1$ it was shown that $\Delta - J = \lambda \frac{a_1}{J} + ...$ and a remarkable agreement was found

between $a_1(\frac{J_1}{J_2})$ and the coefficient c_1 in the energies of various 2-spin string solutions. As in the BMN case, it was possible also to match the energies of fluctuations near the circular $J_1 = J_2$ solution with the corresponding eigenvalues of (6) [29].

Similar leading-order agreement between string energies and SYM dimensions was observed also in other sectors of states with large quantum numbers: (i) for specific solutions [22, 25, 26] in the $SU(3)$ sector with 3 spins in S^5 dual to $\mathrm{tr}(\Phi_1^{J_1}\Phi_2^{J_2}\Phi_3^{J_3}) + ...$ operators [32, 37]; (ii) for a folded string state [6] belonging to the $SL(2)$ [38] sector with one spin in AdS_5 and one spin in S^5 (with $E = J + S + \frac{\lambda}{J}c_1(\frac{S}{J}) + ...$ [6, 22]) dual to $\mathrm{tr}(D^S\Phi^J) + ...$ [30]; (iii) in a "subsector" of $SO(6)$ states containing pulsating (and rotating) solutions [29, 32] which again have regular energy expansion in the limit of large oscillation number, e.g., $E = L + c_1\frac{\lambda}{L} + ...$ [39].

2 Effective actions for coherent states

The observed agreement between energies of particular semiclassical string states and dimensions of the corresponding "long" SYM operators leaves many questions, in particular: (i) How to understand this agreement beyond specific examples, i.e. in a universal way? (ii) Which is the precise relation between profiles of string solutions and the structure of the dual SYM operators? (iii) How to characterise the set of semiclassical string states and dual SYM operators for which the correspondence should work? (iv) Why agreement works, i.e. why the two limits (first $J \to \infty$, and then $\tilde{\lambda} \to 0$, or vice versa) taken on the string and SYM sides give equivalent results? Should it work to all orders in expansion in $\tilde{\lambda}$ (and $\frac{1}{J}$)? The questions (i),(ii) were addressed in [33, 35, 42, 43, 44]; an alternative approach based on matching the general solution (and integrable structure) of the string sigma model with that of the thermodynamic limit of the Bethe ansatz was developed in [34]. The question (iii) was addressed in [45, 46, 47, 44], and the question (iv) – in [15, 16, 17].

One key idea of [33] (elaborated further in [35, 44]) was that instead of comparing particular solutions one should try to match effective sigma models which appear on the string side and the SYM side. Another related idea of [33, 35, 44] was that since "semiclassical" string states carrying large quantum numbers are represented in the quantum theory by coherent states, one should be comparing coherent string states to coherent SYM states (i.e. to coherent states of the spin chain). Because of the ferromagnetic nature of the dilatation operator (6) in the thermodynamic limit $J = J_1 + J_2 \to \infty$ with fixed number of impurities $\frac{J_1}{J_2}$ it is favorable to form large clusters of spins and thus a "low-energy"

approximation and continuum limit apply, leading to an effective "non-relativistic" sigma model for a coherent-state expectation value of the spin operator. Taking the "large energy" limit directly in the string action gives a reduced "non-relativistic" sigma model that describes in a universal way the leading-order $O(\tilde{\lambda})$ corrections to the energies of all string solutions in the two-spin sector. The resulting action agrees exactly [33] with the semiclassical coherent state action describing the $SU(2)$ sector of the spin chain in the $J \to \infty$, $\tilde{\lambda}$ =fixed limit. This demonstrates how a string action can directly emerge from a gauge theory in the large-N limit and provides a direct map between "coherent" SYM states (or operators built out of two holomorphic scalars) and all two-spin classical string states. Furthermore, the correspondence established at the level of the action implies also the matching of fluctuations around particular solutions (as in the BMN case) and thus goes beyond the special examples of rigidly rotating strings.

Let us briefly review the definition of coherent states (see, e.g., [40]). For a harmonic oscillator ($[a, a^\dagger] = 1$) one can define the coherent state as $|u\rangle$ as $a|u\rangle = u|u\rangle$, where u is a complex number. Equivalently, $|u\rangle = R(u)|0\rangle$, where $R = e^{ua^\dagger - u^* a}$ so that acting on $|0\rangle$ R is simply proportional to $e^{ua^\dagger}$. Note that $|u\rangle$ can be written as a superposition of eigenstates $|n\rangle$ of the harmonic oscillator Hamiltonian, $|u\rangle \sim \sum_{n=0}^{\infty} \frac{u^n}{\sqrt{n!}}|n\rangle$. An alternative definition of coherent state is that it is a state with minimal uncertainty for both coordinate $\hat{q} = \frac{1}{\sqrt{2}}(a + a^\dagger)$ and momentum $\hat{p} = -\frac{i}{\sqrt{2}}(a - a^\dagger)$ operators, $\Delta\hat{p}^2 = \Delta\hat{q}^2 = \frac{1}{2}$, $\Delta\hat{p}^2 \equiv \langle u|\hat{p}^2|u\rangle - (\langle u|\hat{p}|u\rangle)^2$. For that reason this is the best approximation to a classical state. If one defines a time-dependent state $|u(t)\rangle = e^{-iHt}|u\rangle$ then the expectation values of $\hat{q}$ and $\hat{p}$ $\langle u|\hat{q}|u\rangle = \frac{1}{\sqrt{2}}(u + u^*)$, $\langle u|\hat{p}|u\rangle = -\frac{i}{\sqrt{2}}(u - u^*)$, will follow the classical trajectories. Starting with the $SU(2)$ algebra $[S_3, S_\pm] = \pm S_\pm$, $[S_+, S_-] = 2S_3$ and considering the $s = 1/2$ representation where $\vec{S} = \frac{1}{2}\vec{\sigma}$ one can define spin coherent state as a linear superposition of spin up and spin down states: $|u\rangle = R(u)|0\rangle$, where $R = e^{uS_+ - u^* S_-}$, $|0\rangle = |\frac{1}{2}, \frac{1}{2}\rangle$ and u is a complex number. An equivalent way to label the coherent state is by a unit 3-vector $\vec{n}$ defining a point of S^2. Then $|\vec{n}\rangle = R(\vec{n})|0\rangle$ where $|0\rangle$ corresponds to a 3-vector $(0, 0, 1)$ along the 3rd axis ($\vec{n} = U^\dagger\vec{\sigma}U$, $U = (u_1, u_2)$) and $R(\vec{n})$ is an $SO(3)$ rotation from a north pole to a generic point of S^2. The key property of the coherent state is that $\vec{n}$ determines the expectation value of the spin operator: $\langle\vec{n}|\vec{S}|\vec{n}\rangle = \frac{1}{2}\vec{n}$.

In general, one can rewrite the usual phase space path integral as an integral over the overcomplete set of coherent states (for the harmonic oscillator this is simply the change of variables $u = \frac{1}{\sqrt{2}}(q + ip)$):

$$Z = \int [du]\ e^{i\mathrm{S}[u]} \ , \qquad \mathrm{S} = \int dt \Big(\langle u| i\frac{d}{dt}|u\rangle - \langle u|H|u\rangle \Big) \ , \tag{7}$$

where the first (WZ or "Berry phase") term is the analog of the usual $p\dot{q}$ term in the

phase-space action. Applying this to the case of the Heisenberg spin chain Hamiltonian (6) one ends up with with the following action for the coherent state variables $\vec{n}_l(t)$ at sites $l = 1, ..., J$ (see also [41]):

$$\mathrm{S} = \int dt \sum_{l=1}^{J} \left[\vec{C}(n_l) \cdot \vec{n}_l - \frac{\lambda}{2(4\pi)^2} (\vec{n}_{l+1} - \vec{n}_l)^2 \right] . \tag{8}$$

Here $dC = \epsilon^{ijk} n_i dn_j \wedge dn_k$ (i.e. $\vec{C}$ is a monopole potential on S^2). In local coordinates (at each l) $\vec{n} = (\sin\theta\ \cos\phi,\ \sin\theta\ \sin\phi,\ \cos\theta)$, $\vec{C} \cdot d\vec{n} = \frac{1}{2} \cos\theta d\phi$. In the limit $J \to \infty$ with fixed $\tilde{\lambda} = \frac{\lambda}{J^2}$ one can take a continuum limit by introducing the 2-d field $\vec{n}(t, \sigma) = \{\vec{n}(t, \frac{2\pi}{J} l)\}$. Then

$$\mathrm{S} = J \int dt \int_0^{2\pi} \frac{d\sigma}{2\pi} \left[\vec{C} \cdot \partial_t \vec{n} - \frac{1}{8} \tilde{\lambda} (\partial_\sigma \vec{n})^2 + ... \right] , \tag{9}$$

where dots stand for higher derivative terms suppressed by $\frac{1}{J}$. In the limit $J \to \infty$ we are interested in all quantum corrections are thus suppressed by $\frac{1}{J}$, and thus the above action can be treated classically. The corresponding equation of motion $\dot{n}_i = \frac{1}{2} \tilde{\lambda} \epsilon_{ijk} n_j n_k''$ are the Landau-Lifshitz equation for a classical ferromagnet.

The action (9) should be describing the coherent states of the Heisenberg spin chain in the above thermodynamic limit. One may wonder how a similar "non-relativistic" action may appear on the string side where one starts with a usual sigma model (1). To obtain such an effective action one is to perform the following procedure [33, 35, 44]: (i) isolate a "fast" coordinate α whose momentum p_α is large in our limit; (ii) gauge fix $t \sim \tau$ and $p_\alpha \sim J$ (or $\tilde{\alpha} \sim \sigma$ where $\tilde{\alpha}$ is "T-dual" to α); (iii) expand the action in derivatives of "slow" or 'transverse" coordinates (to be identified with $\vec{n}$). Let us consider the $SU(2)$ sector of string states carrying two large spins in S^5, with string motions restricted to S^3 part of S^5. The relevant part of the $AdS_5 \times S^5$ metric is then $ds^2 = -dt^2 + d\mathrm{X}_i d\mathrm{X}_i^*$, with $\mathrm{X}_i \mathrm{X}_i^* = 1$. Let us set

$$\mathrm{X}_1 = X_1 + iX_2 = u_1 e^{i\alpha} , \qquad \mathrm{X}_2 = X_3 + iX_4 = u_2 e^{i\alpha} , \qquad u_i u_i^* = 1 .$$

Here α will be a collective coordinate associated to the total (large) spin in the two planes (which in general will be the sum of orbital and internal spin); u_i (defined modulo $U(1)$ gauge transformation) will be the "slow" coordinates determining the "transverse" string profile. Then

$$d\mathrm{X}_i d\mathrm{X}_i^* = (d\alpha + C)^2 + Du_i Du_i^* , \qquad C = -iu_i^* du_i , \qquad Du_i = du_i - iCu_i ,$$

and the second term represent the metric of CP^1 (this parametrisation corresponds to Hopf fibration $S^3 \sim S^1 \times S^2$). Introducing $\vec{n} = U^\dagger \vec{\sigma} U$, $U = (u_1, u_2)$ we get $d\mathrm{X}_i d\mathrm{X}_i^* =$

$(D\alpha)^2 + \frac{1}{4}(d\vec{n})^2$, $D\alpha = d\alpha + C(n)$. Writing the resulting sigma model action in phase space form and imposing the (non-conformal) gauge $t = \tau$, p_α =const= J one gets the action (9) with the WZ term $\vec{C} \cdot \partial_t \vec{n}$ originating from the $p_\alpha D\alpha$ term in the phase-space Lagrangian (cf. its origin on the spin chain side as an analog of the '$p\dot{q}$' in the coherent state path integral action). Equivalent approach is based on first applying a 2-d duality (or "T-duality") $\alpha \to \tilde{\alpha}$ and then choosing the "static" gauge $t = \tau$, $\tilde{\alpha} = \frac{1}{\sqrt{\tilde{\lambda}}}\sigma$, $\frac{1}{\sqrt{\tilde{\lambda}}} = \frac{J}{\sqrt{\lambda}}$. Indeed, starting with

$$\mathcal{L} = -\frac{1}{2}\sqrt{-g}g^{pq}(- \partial_p t \partial_q t + D_p\alpha D_q\alpha + D_p u_i^* D_q u_i) \tag{10}$$

and applying T-duality in α we get

$$\mathcal{L} = -\frac{1}{2}\sqrt{-g}g^{pq}(- \partial_p t \partial_q t + \partial_p\tilde{\alpha}\partial_q\tilde{\alpha} + D_p u_i^* D_q u_i) + \epsilon^{pq} C_p \partial_q \tilde{\alpha} \ . \tag{11}$$

Thus the "T-dual" background has no off-diagonal metric component but has a non-trivial NS-NS 2-form coupling in the $(\tilde{\alpha}, u_i)$ sector. It is important that we do not use conformal gauge here. Eliminating the 2-d metric g^{pq} we then get the Nambu-type action

$$\mathcal{L} = \epsilon^{pq} C_p \partial_q \tilde{\alpha} \ - \sqrt{h} \ , \tag{12}$$

where $h = |\det\, h_{pq}|$ and $h_{pq} = -\partial_p t \partial_q t + \partial_p\tilde{\alpha}\partial_q\tilde{\alpha} + D_{(p} u_i^* D_{q)} u_i$. If we now fix the static gauge $t = \tau$, $\tilde{\alpha} = \frac{1}{\sqrt{\tilde{\lambda}}}\sigma$ we finish with the following action $I = J \int dt \int_0^{2\pi} \frac{d\sigma}{2\pi}\, \mathcal{L}$, where

$$\mathcal{L} = C_0 - \sqrt{(1 + \tilde{\lambda}|D_1 u_i|^2)(1 - |D_0 u_i|^2) + \frac{1}{4}\tilde{\lambda}(D_0 u_i^* D_1 u_i + c.c.)^2} \ . \tag{13}$$

To leading order in $\tilde{\lambda}$ this gives

$$\mathcal{L} = -i u_i^* \partial_0 u_i - \frac{1}{2}\tilde{\lambda}|D_1 u_i|^2 \ , \tag{14}$$

which is the same as the CP^1 Landau-Lifshitz action (9) when written in terms of $\vec{n}$. Thus the string-theory counterpart of the WZ term in the spin-chain coherent state effective action comes from the 2-d NS-NS WZ term upon the static gauge fixing in the "T-dual" $\tilde{\alpha}$ action [44].

To summarize: (i) $(t, \tilde{\alpha})$ are the "longitudinal" coordinates that are gauge-fixed (with $\tilde{\alpha}$ playing the role of string direction or spin chain direction on the SYM side); (ii) $U = (u_1, u_2)$ or $\vec{n} = U^\dagger \vec{\sigma} U$ are "transverse" coordinates that determine the semiclassical string profile and also the structure of the coherent operator on the SYM side, tr $\Pi_\sigma(u_i \Phi_i)$. The agreement between the low-energy actions on the spin chain and the string side explains not only the matching of energies of coherent states for configurations with two large

spins (and near-by fluctuations) but also the matching of integrable structures observed on specific examples in [31, 32].

This leading-order agreement in $SU(2)$ sector has several generalizations. First, we may include higher-order terms on the string side. Expanding (13) in $\tilde{\lambda}$ and eliminating higher powers of time derivatives by field redefinitions (note that leading-order equation of motion is 1-st order in time derivative) we end up with [35]

$$\mathcal{L} = \vec{C} \cdot \dot{\vec{n}} - \frac{\tilde{\lambda}}{8}\vec{n}'^2 + \frac{\tilde{\lambda}^2}{32}(\vec{n}''^2 - \frac{3}{4}\vec{n}'^4) - \frac{\tilde{\lambda}^3}{64}[\vec{n}'''^2 - \frac{7}{4}\vec{n}'^2\vec{n}''^2 - \frac{25}{2}(\vec{n}'\vec{n}'')^2 + \frac{13}{16}\vec{n}'^6] + ...$$

The same $\tilde{\lambda}^2$ term is obtained [35] in the coherent state action on the spin chain side by starting with the sum of the 1-loop dilatation operator (6) and the 2-loop term found in [11]

$$D_2 = \frac{\lambda^2}{(4\pi)^4} \sum_{l=1}^{J} (\mathrm{Q}_{l,l+2} - 4\mathrm{Q}_{l,l+1}) \ , \qquad \mathrm{Q}_{k,l} \equiv I - \vec{\sigma}_k \cdot \vec{\sigma}_l \ . \tag{15}$$

This explains the matching of energies and dimensions to the first two orders, as first observed on specific examples using Bethe ansatz in [15]. The equivalent general conclusion about 2-loop matching was obtained in the integrability-based approach in [34]. The order-by-order agreement seems to break down at $\tilde{\lambda}^3$ (3-loop) order, and a natural reason [15, 16] is that the string limit (first $J \to \infty$, then $\tilde{\lambda} \to 0$) and the SYM limit (first $\lambda \to 0$, then $J \to \infty$) need not be the same. Suggestions how to "complete" the gauge-theory answer to have the agreement with string theory appeared in [16, 17].[2]

One can also generalize [42, 43] the above leading-order agreement to the $SU(3)$ sector of states with three large S^5 spins J_i, $i = 1, 2, 3$, finding the CP^2 analog of the CP^1 "Landau-Lifshitz" Lagrangian in (9),(14) [43] $\mathcal{L} = -iu_i^* \partial_0 u_i - \frac{1}{2}\tilde{\lambda}|D_1 u_i|^2$ on both string and spin chain sides. Similar conclusion is reached [43] in the $SL(2)$ sector of (S, J) states (using the dilatation operator of [38]), where $\vec{n} \to \vec{l}$, $l_1^2 - l_2^2 - l_3^2 = 1$. Finally, one can consider also pulsating string states discussed in the next section.

[2] They also resolve the order $\tilde{\lambda}^3$ disagreement [9] between string and gauge theory predictions for $\frac{1}{J}$ corrections to the BMN spectrum. A possible explanation of why the agreement took place at first two orders in $\tilde{\lambda}$ is that the structure of the dilatation operator at one and two loop order is, in a sense, fixed by the BMN limit, which thus essentially determines the low energy effective action in a unique way; this is no longer so starting with 3-loop order.

3 General fast motion in S^5 and scalar operators from $SO(6)$ sector

One would like to try to understand the general conditions on string states and SYM operators for which the above correspondence works, and, in particular, incorporate also states with large oscillation numbers. Here we will follow [43, 44] (a closely related approach was developed in [46, 47]). For strings moving in S^5 with large oscillation number $E = L + c_1 \frac{\lambda}{L} + ...$, i.e. the limit $L \to \infty$, $\tilde{\lambda} = \frac{\lambda}{L^2} \to 0$ is again regular [39], and the leading-order duality relation between string energies and anomalous dimensions was checked in [29, 32, 48]. The general condition on string solutions for which $E/L = f(\tilde{\lambda})$ has a regular expansion in $\tilde{\lambda}$ appears to be that the world sheet metric should degenerate [46] in the $\tilde{\lambda} \to 0$ limit, i.e. the string motion should be ultra-relativistic in the small string tension limit [45]. In the strict tensionless $\tilde{\lambda} \to 0$ limit each string piece is following a geodesic (big circle) of S^5, while switching on tension leads to a slight deviation from geodesic flow, i.e. to a nearly-null world surface [46]. The dual coherent SYM operators are then "locally BPS", i.e. each string bit corresponds to a BPS linear combination of 6 scalars. In general, the scalar operators can be written as

$$\mathcal{O} = C_{m_1...m_L} \mathrm{tr}(\phi_{m_1}...\phi_{m_L}) \ .$$

The planar 1-loop dilatation operator acting on $C_{m_1...m_L}$ was found in [28] (and is equivalent to an integrable $SO(6)$ spin chain Hamiltonian)

$$H^{n_1...n_L}_{m_1\cdots m_L} = \frac{\lambda}{(4\pi)^2} \sum_{l=1}^{L} \Big(\delta_{m_l m_{l+1}} \delta^{n_l n_{l+1}} + 2\delta^{n_l}_{m_l} \delta^{n_{l+1}}_{m_{l+1}} - 2\delta^{n_{l+1}}_{m_l} \delta^{n_l}_{m_{l+1}} \Big) \ . \tag{16}$$

To find the analog of the coherent-state action (14) we choose a natural set of coherent states $\Pi_l |v_l\rangle$, where at each site $|v\rangle = R(v)|0\rangle$. Here R is an $SO(6)$ rotation and $|0\rangle$ is the BPS vacuum state corresponding to $\mathrm{tr}(\phi_1 + i\phi_2)^L$ or $v_{(0)} = (1, i, 0, 0, 0, 0)$, which is invariant under $H = SO(2) \times SO(4)$. Then the rotation $R(v)$ and thus the coherent state is parametrized by a point in $G/H = SO(6)/[SO(4) \times SO(2)]$, i.e. v belongs to the Grassmanian $G_{2,6}$ [43]. $G_{2,6}$ is thus the coherent state target space for the spin chain sigma model since it parametrizes the orbits of the half-BPS operator $\phi_1 + i\phi_2$ under the $SO(6)$ rotations. This is the space of 2-planes passing through zero in R^6, or the space of big circles in S^5, i.e. the moduli space of geodesics in S^5 [47]. It can be represented also as an 8-dimensional quadric in CP^5: a complex 6-vector v_m should be subject, in addition to $v_m v^*_m = 1$ (and gauging away the common phase) also to $v_m v_m = 0$ condition. Taking the limit $L \to \infty$ with fixed $\tilde{\lambda} = \frac{\lambda}{L^2}$ and the continuum limit $v_{lm}(t) \to v_m(t, \sigma)$ we

then get the $G_{2,6}$ analog of the CP^1 action (9),(14)

$$S = L \int dt \int_0^{2\pi} \frac{d\sigma}{2\pi} \left(-iv_m^* \partial_0 v_m - \frac{1}{2}\tilde{\lambda}|D_1 v_m|^2 \right) , \qquad v_m v_m^* = 1 , \quad v_m v_m = 0 , \quad (17)$$

where as in CP^n case $D_1 v_m = \partial_1 v_m - (v^* \partial_1 v) v_m$.

One may wonder how this 8-dimensional sigma model can be related to string sigma model on $R \times S^5$ where the coordinate space of transverse motions is only 4-dimensional. The crucial point is that the coherent state action is defined on the phase space (cf. the harmonic oscillator case), and $8 = (1+5) \times 2 - 2 \times 2$ is indeed the phase space dimension of a string moving in S^5. On the string side, the need to use the phase space description is related to the fact that to isolate a "fast" coordinate α for a generic string motion we need to specify both the position and velocity of each string piece. Given $\mathcal{L} = -(\partial t)^2 + (\partial X_m)^2$ in conformal gauge ($\dot{X}X' = 0,\ \dot{X}^2 + X'^2 = \kappa^2,\ X_m^2 = 1$) we find the geodesics as $X_m = a_m \cos\alpha + b_m \sin\alpha$, where $\alpha = \kappa\tau,\ a_m^2 = 1,\ b_m^2 = 1,\ a_m b_m = 0$. Equivalently, $X_m = \frac{1}{\sqrt{2}}(e^{i\alpha} v_m + e^{-i\alpha} v_m^*)$, where $v_m = \frac{1}{\sqrt{2}}(a_m - i b_m),\ v_m v_m^* = 1,\ v_m v_m = 0$, i.e. the constant v_m belongs to $G_{2,6}$. In general, for near-relativistic string motions v_m should change slowly with τ and σ. Then starting with the phase space Lagrangian for (X_m, p_m)

$$L = p_m \dot{X}_m - \frac{1}{2} p_m p_m - \frac{1}{2} X'_m X'_m$$

we may change the variables according to [44] (cf. harmonic oscillator case)

$$X_m = \frac{1}{\sqrt{2}}(e^{i\alpha} v_m + e^{-i\alpha} v_m^*) , \qquad p_m = \frac{i}{\sqrt{2}} p_\alpha (e^{i\alpha} v_m - e^{-i\alpha} v_m^*) , \quad (18)$$

where α and v_m now depend on τ and σ and v_m belongs to $G_{2,6}$. There is an obvious $U(1)$ gauge invariance, $\alpha \to \alpha - \beta,\ v_m \to e^{i\beta} v_m$. Gauge-fixing $t \sim \tau,\ p_\alpha \sim L$ (or, after approximate T-duality in α, $\tilde{\alpha} \sim \sigma$) one finds that the phase-space Lagrangian becomes (after a rescaling of the time coordinate) [44]:

$$\mathcal{L} \sim p_\alpha D_0 \alpha - \frac{1}{2}\tilde{\lambda}|D_1 v|^2 - \frac{1}{4}\tilde{\lambda}[e^{2i\alpha}(D_1 v)^2 + c.c.] . \quad (19)$$

The first term here produces $-iv_m^* \partial_0 v_m$ and the last term averages to zero since $\alpha \approx \kappa\tau + ...$ where $\kappa = (\sqrt{\tilde{\lambda}})^{-1} \to \infty$. Equivalently, the α-dependent terms in the action (that were absent in the pure-rotation $SU(3)$ sector) can be eliminated by canonical transformations [44]. We then end up with the following 8-dimensional phase-space Lagrangian for the "transverse" string motions: $\mathcal{L} = -iv_m^* \partial_0 v_m - \frac{1}{2}\tilde{\lambda}|D_1 v_m|^2$, which is the same as found on the spin chain side (17). The 3-spin $SU(3)$ rotation case is the special case when $v_m = (u_1, iu_1, u_2, iu_2, u_3, iu_3)$, where u_i belongs to CP^2 subspace of $G_{2,6}$. The agreement

between the spin chain and the string sides in this general $G_{2,6} = SO(6)/[SO(4)\times SO(2)]$ case explains not only the matching for pulsating solutions [39, 32] but also for near-by fluctuations.

Let us now discuss the reason for the restriction $v^2 = 0$ and also the structure of coherent operators corresponding to semiclassical string states. Given $\mathcal{O} = C_{m_1\ldots m_L}\mathrm{tr}\,(\phi_{m_1}\ldots\phi_{m_L})$ one may obtain the Schrödinger equation for the wave function $C(t)$ from[3]

$$S = -\int dt \Big(iC^*_{m_1\ldots m_L}\frac{d}{dt}C_{m_1\ldots m_L} + C^*_{m_1\ldots m_L}H^{n_1\ldots n_L}_{m_1\cdots m_L}C_{n_1\ldots n_L}\Big) \ . \tag{20}$$

In the limit $L\to\infty$ we may consider the coherent state description and assume the factorized ansatz

$$C_{m_1\ldots m_L} = v_{m_1}\ldots v_{m_L} \ ,$$

where each v_{ml} is a complex unit-norm vector [44]. The BPS case corresponding to totally symmetric traceless $C_{m_1\ldots m_L}$ is represented by $v_{ml} = v_{m(0)}$, $v^2_{(0)} = 0$. Using (16) and substituting the ansatz for C into the above action one finds

$$S = -\int dt \sum_{l=1}^{L}\Big(iv^*_l\frac{d}{dt}v_l + \frac{\lambda}{(4\pi)^2}\Big[(v^*_lv^*_{l+1})(v_lv_{l+1}) + 2 - 2(v^*_lv_{l+1})(v_lv^*_{l+1})\Big]\Big) \ . \tag{21}$$

As expected [28], the Hamiltonian (second term) vanishes for the BPS case when v_l does not depend on l and $v^2 = 0$. More generally, if we assume that v_l is changing slowly with l (*i.e.* $v_l \simeq v_{l+1}$), then we find that (21) contains a potential term $(v^*_lv^*_l)(v_lv_l)$ coming from the first "trace" structure in (16). This term will lead to large (order λL [28]) shifts of anomalous dimensions, invalidating low-energy expansion, i.e. prohibiting one from taking the continuum limit $L\to\infty$, $\tilde{\lambda} = \frac{\lambda}{L^2} =$ fixed, and thus from establishing correspondence with string theory along the lines of [33, 35, 43]. To get solutions with variations of v_l from site to site small we are to impose $v^2_l = 0$, $l = 1,\ldots,L$ which minimizes the potential energy coming from the first term in (16). This condition implies that the operator at each site is invariant under half of supersymmetries: if $v^2 = 0$ the matrix $v_m\Gamma^m$ appearing in the variation of the operator $v_m\phi_m$, i.e. $\delta_\epsilon(v_m\phi_m) = \frac{i}{2}\bar{\epsilon}(v_m\Gamma^m)\psi$, satisfies $(v_m\Gamma^m)^2 = 0$. This means that $v_m\phi_m$ is invariant under the variations associated with the null eigenvalues. One may thus call $v^2 = 0$ a "*local BPS*" condition since the preserved combinations of supercharges in general are different for each v_l, i.e. the operator corresponding to $C = v_1\ldots v_L$ is not BPS. Here "local" should be understood in

[3]For coherent states we consider the equation of motion that follows from this action may be interpreted as a (non-trivial) RG equation for the coupling constant associated to the operator $\mathcal{O}$.

the sense of the spin chain, or, equivalently, the spatial world-sheet direction.[4] In the case when v_l are slowly changing we can take the continuum limit as in [33, 35, 43] by introducing the 2-d field $v_m(t,\sigma)$ with $v_{ml}(t) = v_m(t, \frac{2\pi l}{L})$. Then (21) reduces to (17) (all higher derivative terms are suppressed by powers of $\frac{1}{L}$ and the potential term is absent due to the condition $v^2 = 0$), i.e. (21) becomes equivalent to the $G_{2,6}$ "Landau-Lifshitz" sigma model which was derived from the phase space action on the string side. [5]

To summarize, considering ultra-relativistic strings in S^5 one can isolate a fast variable α (a "polar angle" in the string phase space) whose momentum p_α is large. One may gauge fix p_α to be constant $\sim L$, or $\tilde{\alpha} \sim L\sigma$, so that σ or the "operator direction" on the SYM side gets interpretation of "T-dual to fast coordinate" direction. As a result, one finds a local phase-space action with 8-dimensional target space (where one no longer can eliminate 4 momenta without spoiling the locality). This action is equivalent to the Grassmanian $G_{2,6}$ Landau-Lifshitz sigma model action appearing on the spin chain side.

We thus get a precise mapping between string solutions and operators representing coherent spin chain states. Explicit examples corresponding to pulsating and rotating solutions are given in [44]. In the continuum limit we may write the operator corresponding to the solution $v(t,\sigma)$ as $\mathcal{O} = \mathrm{tr}(\prod_\sigma \mathrm{v}(t,\sigma)),\ \mathrm{v} \equiv v_m(t,\sigma)\phi_m$. This locally BPS coherent operator is the SYM operator naturally associated to a ultra-relativistic string solution. The t-dependence of the string solution thus translates into the RG scale dependence of $\mathcal{O}$, while the σ-dependence describes the ordering of the factors under the trace.

In general, semiclassical string states represented by classical string solutions should be dual to coherent spin chain states or coherent operators, which are different from the exact eigenstates of the dilatation operator but which should lead to the same energy or anomalous dimension expressions. At the same time, the Bethe ansatz approach [28, 29, 30, 34] is determining the exact eigenvalues of the dilatation operator. The reason why the two approaches happen to be in agreement is that in the limit we consider the problem is essentially semiclassical, and because of the integrability of the spin chain, its exact eigenvalues are not just well-approximated by the classical solutions but are actually exactly reproduced by them, i.e. (just as in the harmonic oscillator or flat space string

[4]This generalizes the argument implicit in [33]; an equivalent proposal was made in [47]. This is related to but different from the "nearly BPS" operators discussed in [45] (which, by definition, were those which become BPS in the limit $\lambda \to 0$).

[5]The presence of the trace in the SYM operators implies that we have to consider only spin chain states that are invariant under translations in l or in σ. This means that the momentum in the direction σ should vanish: $P_\sigma = 0$, or, equivalently, $\int_0^{2\pi} \frac{d\sigma}{2\pi}\ v_m^* \partial_\sigma v_m = 0$. This should be viewed as a condition on the solutions $v_m(t,\sigma)$. The same condition appears on the string side from a constraint.

theory case) the semiclassical coherent state sigma model approach happens to be exact.

4 Concluding remarks

As reviewed above, during the last year and a half it was realized that there exists a remarkable generalization of the near-BPS (BMN) limit to non-BPS but "locally-BPS" sector of string/SYM states. This is an important progress in understanding of gauge-string duality at the quantitative level. The hope is to use this to find the string/SYM spectrum exactly, at least in a subsector of states. The relation between phase-space action for "slow" variables on the string side and the coherent-state action on the SYM (spin chain) side gives a very explicit picture of how string action "emerges" from the gauge theory (dilatation operator). It implies not only the equivalence between string energies and SYM dimensions (established to first two orders in expansion in effective coupling $\tilde{\lambda}$) but also a direct relation between the string profiles and the structure of coherent SYM operators [33, 44].

One may try also to use the duality as a tool to uncover the structure of planar SYM theory to all orders in λ by imposing the exact agreement with particular string solutions. For example, demanding the consistency with the BMN scaling limit (along with the superconformal algebra) determines the structure of the full 3-loop SYM dilatation operator in the $SU(2)$ sector [11, 12]. One can also use the BMN limit to fix only a part of the dilatation operator but to all orders in λ [49]. Generalizing (6),(15) and the 3- and 4-loop expressions in [11, 12] one can organize [15, 35, 49] the dilatation operator as an expansion in powers of $\mathrm{Q}_{k,l} = I - \vec{\sigma}_k \cdot \vec{\sigma}_l$ which reflect interactions between spin chain sites,

$$D = \sum \mathrm{Q} + \sum \mathrm{QQ} + \sum \mathrm{QQQ} + \ldots .$$

Here the products Q...Q are "irreducible", i.e. each site index appears only once. The Q^2-terms first appear at 3 loops, Q^3-terms – at 5 loops, etc. [11, 12]. Concentrating on the order Q part $D^{(1)}$ of D one can write (here $L = J$):

$$D^{(1)} = \sum_{r=0}^{\infty} \frac{\lambda^r}{(4\pi)^r} \sum_{l=1}^{L} \mathcal{D}_r(l) , \qquad \mathcal{D}_r(l) = \sum_{k=1}^{r} a_{r,k} \mathrm{Q}_{l,l+k} , \tag{22}$$

or $D^{(1)} = \sum_{l=1}^{L} \sum_{k=1}^{L-1} h_k(L, \lambda)\, \mathrm{Q}_{l,l+k}$. Demanding the agreement with the BMN limit one can then determine the coefficients $a_{r,k}$ and thus the function h_k explicitly to all orders

in λ [49]. In particular, for large L, i.e. when D acts on "long" operators, one finds

$$D^{(1)} = \sum_{l=1}^{L} \sum_{k=1}^{\infty} f_k(\lambda)\ \mathrm{Q}_{l,l+k}\ , \qquad\qquad f_k(\lambda) = \sum_{r=k}^{\infty} \frac{\lambda^r}{(4\pi)^{2r}}\ a_{r,l}\ , \tag{23}$$

where the coefficients $f_k(\lambda)$ can be summed up in terms of the standard Gauss hypergeometric function [49]

$$f_k(\lambda) = \left(\frac{\lambda}{4\pi^2}\right)^k \frac{\Gamma(k-\frac{1}{2})}{4\sqrt{\pi}\ \Gamma(k+1)}\ \ {}_2F_1(k-\frac{1}{2}, k+\frac{1}{2}; 2k+1; -\frac{\lambda}{\pi^2})\ . \tag{24}$$

The function $f_k(\lambda)$ smoothly interpolates between the usual perturbative expansion at small λ and $f_k(\lambda) \sim \sqrt{\lambda}$ at strong λ (which is the expected behaviour of anomalous dimensions of "long" operators dual to "semiclassical" states). Similar interpolating functions are expected to appear in anomalous dimensions of other SYM operators. Also, f_k goes rapidly to zero at large k, so we get a spin chain with short range interactions.

One may hope that imposing additional constraints coming from correspondence with other string solutions may help to determine the dilatation operator further (see also [16, 17]).

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